Appendix C: Generation of Benchmark Buildings

In order to present details of the most useful rocking wall size for benchmark buildings, those benchmark buildings must be defined.

There are many ways in which this could be done. The method that seems most natural is to start with basic assumptions that a structural engineer would have about a building of a given size. One of the earliest appreciations that an engineer has is of the natural period of a building.

The natural period may be estimated with the familiar $T \approx 0.1N$. However, that is a very rough approximation, and can be considerably off, especially for certain kinds of structure. A better approach to approximate the period is to follow the guidelines in a widely accepted code, such as *ASCE 7-05*. *ASCE 7-05* considers almost all building types when calculating the natural period, and although it usually produces a result within 25% of $T \approx 0.1N$, is not constrained by the assumptions in that formula.

In addition to story masses, the only other piece of information we need to determine the story stiffnesses is the stiffness form. For example, we need to know whether the story stiffnesses are uniform, or increase linearly from the top down, or increase parabolically from the top down. We cannot model every possible building, but rather hope to determine a representative set of models, one of which will offer a reasonable approximation to most rocking wall installations, and thus provide data which simplifies the analysis process.

It is important to understand that if, in addition to the mass distribution, the stiffness form (constant/linear/parabolic) is known, and code is used to determine the natural period, then the code fully implies the stiffness matrix. Developing a stiffness matrix in addition to using code to determine the natural period is redundant. Likewise, if the stiffness matrix is known, then using the code to find the period is redundant. And since code provides a very useful approximation of the natural period of buildings, it may also be extended to provide a useful set of representative building models. Making the leap from natural period to stiffness matrix, given the stiffness form, requires some careful algebra, which is presented here.

Based on the mass, mass form, and stiffness form, we may apply algebra that rearranges the Rayleigh quotient formula, including inverse iteration, to provide the building stiffnesses, to determine the story stiffnesses, and thus produce a set of representative discretized building models. We will also use an example to suggest that using five inverse iterations is equivalent for any reasonable purpose to a closed-form solution.

NB: Unlike the paper itself, the convention used in this appendix (C) is that the top node of a building is node 1, while the lowest node is node N. This convention is sometimes used in vibrations, although it is not generally conventional structural engineering. This feature will be updated.

In the case where the story stiffnesses are uniform, the Rayleigh quotient is not required, since we may apply the closed-form solution:

$$\omega_1^2 = 4\frac{k}{m}\sin^2\frac{\pi}{4N} \qquad (2.28)^{(2,p139)}$$

where N is the number of stories, m is the story mass of a standard top-light building model, and k is the story stiffness.

2.4.1 Using the Rayleigh Quotient to Find the Building Stiffness

In order to find the stiffness from the natural period of a building, the logical formula to use is the Rayleigh quotient.

When the Rayleigh quotient is used with buildings, a very commonly used initial vector is the linear vector, $\underline{v} = [N, N-1, \ldots, 1]^T$.

If we substitute this vector into the denominator of the Rayleigh quotient formula, with a standard top-light mass matrix $\boldsymbol{M}$, it can be seen that:

$$\underline{v}^T\boldsymbol{M}\underline{v} = m\frac{N}{6}(2N^2 + 1) \qquad (2.29)^{(2,p138)}$$

where

$$\boldsymbol{M} = \begin{bmatrix} \frac{m}{2} & 0 & \cdots & 0 \\ 0 & m & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & m \end{bmatrix} \qquad (2.30)$$

For the numerator, we can apply a similar method in the case where k_i are constant:

$$\underline{v}^T \boldsymbol{K} \underline{v} = Nk \qquad (2.31)^{(2,p138)}$$

However, most buildings do not have a constant k_i, but rather are stiffer at the base, and become less stiff towards the top. Thus it would be useful to extend this idea to other distributions of k_i.

Let us define a matrix $\boldsymbol{S_{D/L}}$, that shifts elements down or left, and $\boldsymbol{S_{U/R}}$, that shifts elements up or right, depending respectively whether they pre- or post-multiply:

$$\boldsymbol{S_{D/L}} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad (2.32)$$

$$\boldsymbol{S_{U/R}} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (2.33)$$

we can see that

$$\underline{v}^T(\boldsymbol{I} - \boldsymbol{S_{D/L}}) = \underline{e}^T \qquad (2.34)$$

$$(\boldsymbol{I} - \boldsymbol{S_{U/R}})\underline{v} = \underline{e} \qquad (2.35)$$

where

$$\underline{e} = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \qquad (2.36)$$

If we study the output of the formula $\underline{v}^T \boldsymbol{K} \underline{v}$, it can be observed that

$$\underline{v}^T \boldsymbol{K} \underline{v} = \sum_{i=1}^{N} k_i \qquad (2.37)$$

This is very useful, and we may use this with the uniterated Rayleigh quotient formula to determine k_i, for some predictable distribution of stiffness, given the natural period of a structure. However, it would be preferable to apply inverse iteration in the solution.

2.4.2 Using Inverse Iteration to Find the Building Stiffness

The next step is to incorporate inverse iteration to the method, to ensure that the values found for k_i are as close as possible to the true values.

However, the Rayleigh quotient formula only allows us to solve for one variable, so in order to use inverse iteration, we must formulate a stiffness matrix specifically for a given arrangement of stiffnesses, in terms of a fixed variable that defines the stiffness distribution, such as k_1.

When each story stiffness is a successive multiple of the top stiffness (i.e. in the following example, the increment $d=1$), the structure deforms linearly, under uniform loading.[3,p58] Let us consider the general linearly-increasing case, in which the increment d may be less than or greater than unity:

$$k_i = k_1 + (j-1)dk_1 \qquad (1 \le j < N;\ d \in \mathbb{R}^+) \qquad (2.38)$$

We would like to solve the following for k_1, or some other variable that uniquely specifies the stiffness matrix, given the stiffness arrangement:

$$R_i = \frac{\underline{v}^T((K^{-1}M)^T)^n K(K^{-1}M)^n \underline{v}}{\underline{v}^T((K^{-1}M)^T)^n M(K^{-1}M)^n \underline{v}} \tag{2.39}$$

Since we know that

$$\underline{v}^T \boldsymbol{K} \underline{v} = \sum_{i=1}^{N} k_i \tag{2.40}$$

and by definition

$$\underline{e}^T \underline{k} = \sum_{i=1}^{N} k_i \tag{2.41}$$

we may equate in the general case

$$\underline{e}^T \underline{k} = \underline{v}^T \boldsymbol{K} \underline{v} \tag{2.42}$$

It would be helpful if we could solve for $\underline{k}$, a vector of story stiffnesses, knowing $\boldsymbol{K}$. If we define

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 \end{bmatrix} \tag{2.43}$$

$$\boldsymbol{U} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.44}$$

we can see that

$$\underline{v} = \boldsymbol{U} \underline{e} \tag{2.45}$$

$$v^T = \underline{e}^T \boldsymbol{L} \tag{2.46}$$

If we also note that

$$(\boldsymbol{I} - \boldsymbol{S}_{D/L}) = \mathbf{L}^{-1} \tag{2.47}$$

$$(\boldsymbol{I} - \boldsymbol{S}_{U/R}) = \mathbf{U}^{-1} \tag{2.48}$$

thus we may rearrange *equation 2.46* for $\underline{e}^T$ and substitute into *equation 2.42*:

$$\underline{v}^T(\boldsymbol{I} - \boldsymbol{S}_{D/L})\underline{k} = \underline{v}^T\boldsymbol{K}\underline{v} \tag{2.49}$$

And so we may observe the following in the general case:

$$\underline{k} = \boldsymbol{LK}\underline{v} \tag{2.50}$$

$$\boldsymbol{LKU} = diag(\underline{k}) \tag{2.51}$$

$$\underline{k} = \boldsymbol{LKU}\underline{e} \tag{2.52}$$

Thus, to find $\boldsymbol{K}$ as a multiple of k_1, or of some other variable that defines a specific arrangement of stiffnesses, we must first define $diag(\underline{k})$ as a multiple of that variable.

2.4.2.1 Linear Stiffness

For the linear case,

$$\underline{k} = k_1 \begin{bmatrix} 1 \\ 1+d \\ 1+2d \\ \vdots \\ 1+(N-1)d \end{bmatrix} \tag{2.53}$$

so

$$diag(\underline{k}) = k_1(\boldsymbol{I} + d\boldsymbol{S}_{D/L}\boldsymbol{F}_l\boldsymbol{V}\boldsymbol{F}_l\boldsymbol{S}_{U/R}) \tag{2.54}$$

where $\boldsymbol{F}_l$ is a ‘flip’ matrix:

$$\boldsymbol{F}_l = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \iddots & 0 \\ 0 & \iddots & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \tag{2.55}$$

where we might note that $\boldsymbol{U} = \boldsymbol{F}_l\boldsymbol{L}\boldsymbol{F}_l$. $\boldsymbol{V}$ is the linear diagonal matrix:

$$\boldsymbol{V} = diag(\underline{v}) = \underline{v}\underline{e}^T.*\boldsymbol{I} = \begin{bmatrix} N & 0 & \cdots & 0 \\ 0 & N-1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \tag{2.56}$$

thus substituting *equation 2.54* into *equation 2.51*,

$$k_1(\boldsymbol{I} + d\boldsymbol{S}_{D/L}\boldsymbol{F}_l\boldsymbol{V}\boldsymbol{F}_l\boldsymbol{S}_{U/R}) = \boldsymbol{LKU} \tag{2.57}$$

rearranging:

$$\boldsymbol{K} = k_1(\boldsymbol{I} - \boldsymbol{S}_{D/L})(\boldsymbol{I} + d\boldsymbol{S}_{D/L}\boldsymbol{F}_l\boldsymbol{V}\boldsymbol{F}_l\boldsymbol{S}_{U/R})(\boldsymbol{I} - \boldsymbol{S}_{U/R}) \tag{2.58}$$

and so

$$\boldsymbol{K}^{-1} = \frac{1}{k_1}\boldsymbol{U}(\boldsymbol{I} + d\boldsymbol{S}_{D/L}\boldsymbol{F}_l\boldsymbol{V}\boldsymbol{F}_l\boldsymbol{S}_{U/R})^{-1}\boldsymbol{L} \tag{2.59}$$

We may thus also substitute this linearly increasing $\boldsymbol{K}$ into the uniterated Rayleigh quotient formula as a simple multiple of k_1:

$$\underline{v}^T\boldsymbol{K}\underline{v} = k_1\underline{v}^T(\boldsymbol{I} - \boldsymbol{S}_{D/L})(\boldsymbol{I} + d\boldsymbol{S}_{D/L}\boldsymbol{F}_l\boldsymbol{V}\boldsymbol{F}_l\boldsymbol{S}_{U/R})(\boldsymbol{I} - \boldsymbol{S}_{U/R})\underline{v} \tag{2.60}$$

simplifying:

$$\underline{v}^T\boldsymbol{K}\underline{v} = \underline{e}^T\underline{k} = k_1\underline{e}^T(\boldsymbol{I} + d\boldsymbol{S}_{D/L}\boldsymbol{F}_l\boldsymbol{V}\boldsymbol{F}_l\boldsymbol{S}_{U/R})\underline{e} \tag{2.61}$$

We may also define the mass matrix in terms of such standard matrices, multiplied by a constant:

$$\boldsymbol{M} = \frac{m}{2}(\boldsymbol{I} + \boldsymbol{S}_{D/L}\boldsymbol{S}_{U/R}) \tag{2.62}$$

and so

$$\boldsymbol{K}^{-1}\boldsymbol{M} = \frac{m}{2k_1}\boldsymbol{U}(\boldsymbol{I} + d\boldsymbol{S}_{D/L}\boldsymbol{F}_l\boldsymbol{V}\boldsymbol{F}_l\boldsymbol{S}_{U/R})^{-1}\boldsymbol{L}(\boldsymbol{I} + \boldsymbol{S}_{D/L}\boldsymbol{S}_{U/R}) \tag{2.63}$$

Thus, in the linear case, the n[th] iteration of the first eigenvector is:

$$\underline{v}_n = (\boldsymbol{K}^{-1}\boldsymbol{M})^n \, \underline{v} = \frac{m^n}{2^n k_1{}^n}[\boldsymbol{U}(\boldsymbol{I} + d\boldsymbol{S}_{D/L}\boldsymbol{F}_l\boldsymbol{V}\boldsymbol{F}_l\boldsymbol{S}_{U/R})^{-1}\boldsymbol{L}(\boldsymbol{I} + \boldsymbol{S}_{D/L}\boldsymbol{S}_{U/R})]^n \, \underline{v} \qquad (2.64)$$

where we may simply choose high n to very closely approximate the true eigenvector. The constants may be neglected of course. Thus we may evaluate the iterated Rayleigh quotient formula for k_1:

$$k_1 = \frac{\omega_1{}^2 \underline{v_n}^T M \underline{v_n}}{\left(\dfrac{\underline{v_n}^T K \underline{v_n}}{k_1}\right)} \qquad (2.65)$$

$$= \frac{\omega_1{}^2 \dfrac{m}{2} \underline{v_n}^T (I + S_{D/L}S_{U/R})\underline{v_n}}{\underline{v_n}^T (I - S_{D/L})(I + dS_{D/L}F_l VF_l S_{U/R})(I - S_{U/R})\underline{v_n}} \qquad (2.66)$$

which we may find in non-dimensional form:

$$\frac{k_1}{m\omega_1{}^2} = \frac{\dfrac{1}{2}\underline{v_n}^T (I + S_{D/L}S_{U/R})\underline{v_n}}{\underline{v_n}^T (I - S_{D/L})(I + dS_{D/L}F_l VF_l S_{U/R})(I - S_{U/R})\underline{v_n}} \qquad (2.67)$$

where $\underline{v}_n$ is found in *equation 2.64*. The constants multiplying $\underline{v}_n$ and $\underline{v}_n{}^T$ are not required.

2.4.2.2 Constant Stiffness

Now that the case of linearly increasing story stiffnesses has illuminated the issues, let us consider the simpler case, where k_i are constant.

We may rearrange *equation 2.28* to find k in the constant case, or we may set $d=0$ in *equation 2.67*. But let us also solve the problem from scratch, in order to compare the results.

For the constant case,

$$diag(\underline{k}) = k\boldsymbol{I} \qquad (2.68)$$

substituting into *equation 2.51*:

$$k\boldsymbol{I} = \boldsymbol{LKU} \qquad (2.69)$$

rearranging:

$$\boldsymbol{K} = k(\boldsymbol{I} - \boldsymbol{S}_{\boldsymbol{D/L}})(\boldsymbol{I} - \boldsymbol{S}_{\boldsymbol{U/R}}) \qquad (2.70)$$

and so

$$\boldsymbol{K}^{\boldsymbol{-1}} = \frac{1}{k}\mathbf{UL} \qquad (2.71)$$

Then if we multiply by the mass matrix defined in *equation 2.62*, then

$$\boldsymbol{K}^{\boldsymbol{-1}}\boldsymbol{M} = \frac{m}{2k}\mathbf{UL}(\boldsymbol{I} + \boldsymbol{S}_{\boldsymbol{D/L}}\boldsymbol{S}_{\boldsymbol{U/R}}) \qquad (2.72)$$

Thus, in the constant case, the nth iteration of the first eigenvector is:

$$\underline{v}_n = (\boldsymbol{K}^{\boldsymbol{-1}}\boldsymbol{M})^n\,\underline{v} = \frac{m^n}{2^n k^n}[\,\mathbf{UL}(\boldsymbol{I} + \boldsymbol{S}_{\boldsymbol{D/L}}\boldsymbol{S}_{\boldsymbol{U/R}}))]^n\,\underline{v} \qquad (2.73)$$

where we may again simply choose high *n* to very closely approximate the true eigenvector, and neglect the constants. Thus we may evaluate the iterated Rayleigh quotient formula for *k*:

$$k = \frac{\omega_1{}^2\,\underline{v}^T M\,\underline{v}}{\left(\dfrac{\underline{v}^T K\,\underline{v}}{k_1}\right)} \qquad (2.74)$$

$$= \frac{\omega_1{}^2\,\dfrac{m}{2}\,\underline{v}^T(I + S_{D/L}S_{U/R})\underline{v}}{\underline{v}^T(I - S_{D/L})(I - S_{U/R})\underline{v}} \qquad (2.75)$$

which matches *equation 2.67* where $d = 0$. We may substitute $\underline{v}=\underline{v}_n$ from *equation 2.73*. The constants multiplying $\underline{v}_n$ and $\underline{v}_n^T$ are not required.

The performance of this formula is excellent as expected. We can compare it with the closed-form solution non-dimensionally:

$$\frac{k}{m\omega_1^2} = \frac{1}{4}\mathrm{cosec}^2\frac{\pi}{4N} \qquad (2.76)^{(2,p139)}$$

For $N = 10$, the closed-form solution yields *40.611909699 to 9 d.p. equation 2.75* yields:

Iterations	$\frac{m\omega_1^2}{k}$	Error (%)
0	33.5000	17.51
1	40.5564955	1.4×10^{-1}
2	40.6112419	1.6×10^{-3}
3	40.6119012	2.1×10^{-5}
4	40.6119096	2.7×10^{-7}
5	40.611909698	3.4×10^{-9}

Table 2.4.2.2.1. *The output of equation 2.75 compared with the closed-form eigenvalue formula*

With each iteration, the error is about two orders of magnitude better. It seems clear that using five iterations would ensure an entirely negligible difference between the estimated value and the true value.

2.4.2.3 Parabolic Stiffness

Since seismic loading is parabolic in nature, a common strategy in building design is to have the stiffness form be parabolic.

It can be seen that if we require constant story drifts δ to ensure that any damage is spread evenly, then

$$\underline{k} = \frac{1}{\delta}\boldsymbol{F_l U}\underline{p} \tag{2.77}$$

where $\underline{k}$ is the stiffness vector, and $\underline{p}$ is any load vector. The inspiration for this neat formula (*equation 2.77*) is Jerome Connor's *Introduction to Structural Motion Control*, which gives a more general form for any desired drift arrangement[3,p58]. It so happens that for constant story drifts, the stiffness vector $\underline{k}$ may be expressed in terms of the standard matrices previously defined.

For equivalent seismic loading, that the loads p_j may be defined as:

$$p_j = \frac{V h_j^k m_j}{\sum_i h_i^k m_i} \qquad (2.78)^{(4,p130)}$$

in which $k = 2$ unless the period is less than $0.5s$, in which case $k = 1$, with the option for linear interpolation between 1 and 2 if the natural period is between $0.5s$ and $2.5s$. For the case $k = 2$, and for the standard top-light mass model, we can express this in vector form as

$$\underline{p} = \frac{Vm}{\sum_i h_i^2 m_i} h_s^2 \begin{bmatrix} \frac{1}{2} & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} N^2 \\ (N-1)^2 \\ \vdots \\ 1^2 \end{bmatrix} \tag{2.79}$$

Where h_s is the single story height. If we denote the constant

$$\frac{Vm}{\sum_i h_i^2 m_i} h_s^2 = A \tag{2.80}$$

then

$$\underline{p} = \frac{1}{2}A(\boldsymbol{I} + \boldsymbol{S_{D/L}S_{U/R}})\begin{bmatrix} N^2 \\ N^2 - 2N + 1^2 \\ N^2 - 4N + 2^2 \\ \vdots \\ \vdots \\ N^2 - 2(N-1)N + (N-1)^2 \end{bmatrix} \tag{2.81}$$

$$= \frac{1}{2}A(\boldsymbol{I} + \boldsymbol{S_{D/L}S_{U/R}})[N^2\underline{e} - \begin{bmatrix} 0 \\ 2 \\ 4 \\ \vdots \\ 2(N-1) \end{bmatrix} N + \begin{bmatrix} 0 \\ 1^2 \\ 2^2 \\ \vdots \\ (N-1)^2 \end{bmatrix}] \tag{2.82}$$

$$= \frac{1}{2}A(\boldsymbol{I} + \boldsymbol{S_{D/L}S_{U/R}})[N^2\underline{e} - 2N\boldsymbol{S_{D/L}F_lU}\underline{e} + \boldsymbol{F_l}(\boldsymbol{S_{U/R}VS_{D/L}})^2\underline{e}] \tag{2.83}$$

$$= \frac{1}{2}A(\boldsymbol{I} + \boldsymbol{S_{D/L}S_{U/R}})[N^2\boldsymbol{I} - 2N\boldsymbol{S_{D/L}F_lU} + \boldsymbol{F_l}(\boldsymbol{S_{U/R}VS_{D/L}})^2]\underline{e} \tag{2.84}$$

where $\boldsymbol{V} = diag(\underline{v})$:

$$\boldsymbol{V} = \begin{bmatrix} N & 0 & \cdots & 0 \\ 0 & N-1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \tag{2.85}$$

And thus substituting *equation 2.84* into *equation 2.77*:

$$\underline{k} = \frac{A}{2\delta}\boldsymbol{F_lU}(\boldsymbol{I} + \boldsymbol{S_{D/L}S_{U/R}})[N^2\boldsymbol{I} - 2N\boldsymbol{S_{D/L}F_lU} + \boldsymbol{F_l}(\boldsymbol{S_{U/R}VS_{D/L}})^2]\underline{e} \tag{2.86}$$

And thus from *equation 2.51*:

$$\boldsymbol{LKU} = diag(\underline{k}) \tag{2.87}$$

It is possible to determine *diag(k̲)* in the parabolic case directly in terms of standard matrices, but the formula is very long, and there is no real benefit to it. Rearranging:

$$\boldsymbol{K} = (\boldsymbol{I} - \boldsymbol{S}_{D/L})diag(\underline{k})(\boldsymbol{I} - \boldsymbol{S}_{U/R}) \tag{2.88}$$

and so

$$\boldsymbol{K}^{-1} = \boldsymbol{U}[diag(\underline{k})]^{-1}\boldsymbol{L} \tag{2.89}$$

Then if we multiply by the mass matrix defined in *equation 2.62*, then

$$\boldsymbol{K}^{-1}\boldsymbol{M} = \frac{m}{2}\boldsymbol{U}[diag(\underline{k})]^{-1}\boldsymbol{L}\,(\boldsymbol{I} + \boldsymbol{S}_{D/L}\boldsymbol{S}_{U/R}) \tag{2.90}$$

Thus, in the parabolic case, the nth iteration of the first eigenvector is:

$$\underline{v}_n = (\boldsymbol{K}^{-1}\boldsymbol{M})^n\,\underline{v}$$

$$= \frac{m^n}{2^n}\,[\boldsymbol{U}[diag(\underline{k})]^{-1}\boldsymbol{L}\,(\boldsymbol{I} + \boldsymbol{S}_{D/L}\boldsymbol{S}_{U/R})]^n\,\underline{v} \tag{2.91}$$

where we may again simply choose high *n* to very closely approximate the true eigenvector, and neglect the constants. Thus we may evaluate the iterated Rayleigh quotient formula, in this case solving for δ, the constant interstory drift:

$$\delta = \frac{\delta\,\underline{v}^T K\,\underline{v}}{\omega_1^{\,2}\,\underline{v}^T M\,\underline{v}} =$$

$$\frac{\delta\,\underline{v}^T(\boldsymbol{I} - \boldsymbol{S}_{D/L})diag(\underline{k})(\boldsymbol{I} - \boldsymbol{S}_{U/R})\underline{v}}{\omega_1^{\,2}\frac{m}{2}\underline{v}^T(\boldsymbol{I} + \boldsymbol{S}_{D/L}\boldsymbol{S}_{U/R})\underline{v}} \tag{2.92}$$

We may substitute $\underline{v}=\underline{v}_n$ from *equation 2.91*. The constants multiplying $\underline{v}_n$ and $\underline{v}_n^{\,T}$ are again not required. Taking $\frac{\delta}{A}$ *to the outside of diag(k̲),* we may now substitute $\frac{\delta}{A}$ into *equation 2.88* to obtain the stiffness matrix.

In summation, the stiffness vector, matrix and mode may be found in any of the three cases with inputs N, w_1, and m, with the addition of d in the linearly increasing case.

An alternative use for *equation 2.92* is to specify the desired story drift δ for a parabolic stiffness arrangement, which will provide constant story drift under equivalent seismic loading, and solve for the resulting period and hence stiffnesses. The same is true for any of the previous stiffness arrangements.